

LOWER BOUNDS IN REAL ALGEBRAIC GEOMETRY AND ORIENTABILITY OF REAL TORIC VARIETIES

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ABSTRACT. The real solutions to a system of sparse polynomial equations may be realized as a fiber of a projection map from a toric subvariety of a sphere. When the toric variety is orientable, the degree of this map is a lower bound for the number of real solutions to the system of equations. Previous work gave conditions which imply that the toric variety is orientable. We strengthen those results by characterizing when the spherical toric variety is orientable. This characterization is based on work of Nakayama and Nishimura, who characterized the orientability of smooth real toric varieties.

INTRODUCTION

A ubiquitous phenomenon in real algebraic geometry is that many geometric problems possess a non-trivial lower bound on their number of real solutions. For example, at least 3 of the 27 lines on a real cubic surface are real as are at least 8 of the 12 rational cubics interpolating 8 real points in the plane [4, Proposition 4.7.3], but there are many, many other examples [1, 6, 7, 9, 15, 14, 17, 18, 21]. While theoretically interesting, this phenomenon has the potential for significant impact on the applications of mathematics as a nontrivial lower bound is an existence proof of real solutions. For this potential to be realized, methods need to be developed to predict when a system of polynomial equations or a geometric problem has a lower bound on its number of real solutions, and to compute or estimate this bound.

We developed a theory of lower bounds on the number of real solutions to systems of sparse polynomials [18]. There, a system of polynomial equations was reformulated as a general fiber of a projection map from a toric subvariety of a sphere. When the smooth points of this toric variety are orientable, the absolute value of the degree of this projection map is a lower bound on the number of real solutions. Besides giving a condition implying this orientability, a method (foldable triangulations of the Newton polytope) was developed to compute the degree of certain maps, and a class of examples of polynomial systems (Wronski polynomial systems from posets) was presented to which this theory applied. Finally, these results were used to give a new proof of the lower bounds in the Schubert calculus of Eremenko and Gabrielov [6].

Further work [10, 11] on foldable triangulations in this context has advanced our understanding of the bound they give. Others [1, 7, 15, 14] have developed additional methods for proving lower bounds in real algebraic geometry and experimentation [17, 18] has revealed many more likely examples of lower bounds.

We give a characterization, in terms of Newton polytopes, of which sparse polynomial systems possess a lower bound in the context of [18]. We do this by extending the work of Nakayama and Nishimura [13], who characterized the orientability of small covers, which

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are topological versions of smooth real projective toric varieties. We characterize the orientability of the smooth points of any (not just smooth and projective) real toric variety, as well as toric subvarieties of a sphere, solving an important open problem from [18].

We review the construction of real toric varieties and spherical toric varieties in Section 1, where we also slightly extend the notion of small covers to not necessarily smooth spaces and formulate our results on orientability. Section 2 contains the mildly technical proof of these results. In Section 3 we use this characterization of orientability to strengthen results from [18] on the theory of lower bounds for the number of real solutions to systems of sparse polynomials.

1. CONSTRUCTIONS OF REAL TORIC VARIETIES

Real toric varieties appear in many applications of mathematics [2, 12, 16] and are interesting objects in their own right [5]. Davis and Januszkiewicz [3] introduced the notion of a small cover of a simple convex polytope as a generalization of smooth projective real toric varieties. We explain a construction of real toric varieties and small covers in terms of the gluing of explicit cell complexes and give a mild extension of Davis and Januszkiewicz's notion of a small cover (which are manifolds) to not necessarily smooth spaces. When a real toric variety is projective, it may be lifted to the sphere which has a two-to-one map to real projective space, and we also describe these spherical toric varieties in terms of the gluing of explicit cell complexes.

Nakayama and Nishimura [13] used this presentation of small covers to characterize their orientability, and we extend their approach to characterize the orientability of the smooth points of the above spaces (not necessarily smooth small covers, nonprojective and spherical toric varieties).

Real toric varieties, singular small covers, and toric subvarieties of the sphere are obtained by gluing the real torus $\mathbb{T}^n := (\mathbb{R}^\times)^n$ or $\{\pm 1\} \times \mathbb{T}^n = \mathbb{T}^{n+1}/\mathbb{R}_\geq$ along copies of \mathbb{T}^{n-1} , one copy for each vector in a collection of integer vectors. There are further gluings and identifications in higher codimension, which presents these spaces as explicit cell complexes. They are smooth at the points of their dense torus \mathbb{T}^n (or at $\{\pm 1\} \times \mathbb{T}^n$) and the attached tori \mathbb{T}^{n-1} , and so their orientability is determined by the gluing along the tori \mathbb{T}^{n-1} . We describe this gluing construction for these different types of spaces.

Complex toric varieties are normal varieties over \mathbb{C} equipped with an action of an algebraic torus $(\mathbb{C}^\times)^n$ having a dense orbit. They are classified by rational fans Σ in \mathbb{R}^n , which encode their construction as a union of affine toric varieties U_σ , one for each cone $\sigma \in \Sigma$. A toric variety may also be viewed as a disjoint union of torus orbits \mathcal{O}_σ , one for each cone $\sigma \in \Sigma$, with the dimension of \mathcal{O}_σ equal to the codimension of the cone σ . The dense orbit \mathcal{O}_0 coincides with the smallest affine patch U_0 , and both are associated to the smallest cone in the fan, the origin 0. See [8] for a complete description.

A toric variety has a canonical set Y of real points which are obtained from the real points of the sets U_σ and orbits \mathcal{O}_σ of the construction. This is described in [8, Ch. 4]. The dense orbit $\mathcal{O}_0(\mathbb{R}) \simeq \mathbb{T}^n$ is isomorphic to $(\mathbb{R}^\times)^n = (\mathbb{R}_{>0})^n \times \{\pm 1\}^n$, which has 2^n components, each a topological n -ball. The subgroup $\{\pm 1\}^n \subset \mathbb{T}^n$ acts on the real toric variety Y , permuting the components of $\mathcal{O}_0(\mathbb{R})$. The orbit space of Y under the group $\{\pm 1\}^n$ is isomorphic to the closure Y_\geq of any component of $\mathcal{O}_0(\mathbb{R})$ in the usual topology (not Zariski!) on Y . Each orbit $\mathcal{O}_\sigma(\mathbb{R})$ has a unique component meeting (and in fact contained in) Y_\geq . We call this component F_σ a *face* of Y_\geq , which is isomorphic to

$(\mathbb{R}_{>0})^{n-\dim(\sigma)}$. Those faces endow Y_{\geq} with the structure of a cell complex that is dual to the fan Σ . That is, the intersection of the closures $\overline{F_{\sigma}}, \overline{F_{\tau}}$ of two faces is the closure $\overline{F_{\rho}}$ of a face where ρ is the minimal cone of Σ containing both σ and τ , and the intersection is empty when there is no cone containing both σ and τ .

The integer points in a cone σ of Σ form a subsemigroup of \mathbb{Z}^n whose image in $(\mathbb{Z}/2\mathbb{Z})^n = \{\pm 1\}^n$ is a subgroup $\overline{\sigma}$ of $\{\pm 1\}^n$. This subgroup $\overline{\sigma}$ is the isotropy subgroup of the face F_{σ} of Y_{\geq} . This gives the following description of Y as a quotient space of $Y_{\geq} \times \{\pm 1\}^n$.

Proposition 1.1. *The real toric variety Y is obtained as the quotient of the cell complex $Y_{\geq} \times \{\pm 1\}^n$ by the equivalence relation where*

$$(p, \xi) \sim (q, \eta) \iff p = q \text{ and } \xi \overline{\sigma} = \eta \overline{\sigma}, \text{ where } p \text{ lies in the face } F_{\sigma}.$$

A *facet* of Y_{\geq} is a face F_{σ} corresponding to a one-dimensional cone σ . The real toric variety Y is smooth at points corresponding to facets, but may not be smooth at points of lower-dimensional faces. If we let Y_{\geq}° be the union of the dense face F_0 and its facets, then

$$Y^{\circ} := (Y_{\geq}^{\circ} \times \{\pm 1\}^n) / \sim$$

consists of smooth points of Y . We study the orientability of Y° (equivalent to that of Y) and its number of connected components.

We generalize this construction. Let P be a finite ranked poset with minimal element 0 and rank at most n where two elements $\sigma, \tau \in P$ either have a unique upper bound in P or else have no upper bound in P . The cones σ in a rational fan in \mathbb{R}^n form such a poset. We also require a cell complex Δ and a system \mathcal{S} of subgroups of $\{\pm 1\}^n$ indexed by P . More specifically, suppose that we have a collection $\mathcal{S} := \{\overline{\sigma} \mid \sigma \in P\}$ of subgroups of $\{\pm 1\}^n$ where $\overline{\sigma} \simeq \{\pm 1\}^{\text{rank}(\sigma)}$, and if $\sigma \subset \tau$, then $\overline{\sigma} \subset \overline{\tau}$. Also suppose that we have a cell complex Δ with cells (called faces) indexed by elements of P ,

$$\Delta = \coprod_{\sigma \in P} F_{\sigma},$$

where each face F_{σ} is a cell of dimension $n - \text{rank}(\sigma)$, which for concreteness, we identify with the interior of the closed unit ball in $\mathbb{R}^{n-\text{rank}(\sigma)}$. We further suppose that:

- Δ is a subset of the closed ball $\overline{F_0}$ in \mathbb{R}^n ,
- the closure of a face F_{σ} in \mathbb{R}^n is homeomorphic to the closed ball of dimension $n - \text{rank}(\sigma)$, and
- given $\sigma, \tau \in P$, the closures of the faces F_{σ} and F_{τ} either do not meet (if σ and τ have no upper bound in P), or their intersection is the closure of the face F_{ρ} , where ρ is the least upper bound of σ and τ in P .

Definition 1.2. Given a cell complex Δ , ranked poset P and system \mathcal{S} of subgroups of $\{\pm 1\}^n$ as above, the *small cover* $Y(\Delta, \mathcal{S})$ of Δ is the quotient

$$(\Delta \times \{\pm 1\}^n) / \sim,$$

where $(p, \xi) \sim (q, \eta)$ if and only if $p = q$ and $\xi \overline{\sigma} = \eta \overline{\sigma}$, where p lies in the face F_{σ} .

Observe that $Y(\Delta, \mathcal{S})$ is equipped with a natural action of $\{\pm 1\}^n$ whose orbit space is Δ , where the orbit of a face F_{σ} is identified with $F_{\sigma} \times \{\pm 1\}^n / \overline{\sigma} \simeq \mathbb{T}^{n-\text{rank}(\sigma)}$. In particular, it is a $\{\pm 1\}^n$ -equivariant compactification of \mathbb{T}^n .

A real toric variety Y associated to a fan Σ is a small cover where P is the set of cones in the fan, $\Delta = Y_{\geq}$, and $\mathcal{S} = \{\bar{\sigma} \mid \sigma \in \Sigma\}$.

The points of $Y(\Delta, \mathcal{S})$ corresponding to the big cell F_0 and to facets F_{σ} are points where $Y(\Delta, \mathcal{S})$ is a topological manifold. Write Δ° for the union of the big cell and the facets, and $Y^{\circ}(\Delta, \mathcal{S}) = (\Delta^{\circ} \times \{\pm 1\}^n) / \sim$ for this subset of the smooth points of $Y(\Delta, \mathcal{S})$.

Real projective toric varieties admit a second construction. Let $\Delta \subset \mathbb{R}^n$ be a full-dimensional polytope with integer vertices, and Σ in \mathbb{R}^n its normal fan. Then the real toric variety Y_{Σ} associated to Σ has a projective embedding given by Δ as follows. Assume that the integer points $\Delta \cap \mathbb{Z}^n$ generate \mathbb{Z}^n . (Otherwise, replace \mathbb{Z}^n by the subgroup generated by $\Delta \cap \mathbb{Z}^n$.) Let \mathbb{P}^{Δ} be the projective space with coordinates indexed by $\Delta \cap \mathbb{Z}^n$ and $y^{\alpha} = y_1^{\alpha_1} \cdots y_n^{\alpha_n}$ the monomial with exponent α . Then we have an injection

$$(1.1) \quad \varphi_{\Delta} : \mathbb{T}^n \ni y \longmapsto [y^{\alpha} \mid \alpha \in \Delta \cap \mathbb{Z}^n],$$

where $[\cdots]$ denotes homogeneous coordinates for \mathbb{P}^{Δ} , where we identify points that are proportional. The closure Y_{Δ} of the image of this map is isomorphic to the real toric variety Y_{Σ} , and the cell complex Y_{\geq}° is identified with the polytope Δ .

The unit sphere $\mathbb{S}^{\Delta} \subset \mathbb{R}^{\Delta}$ has a two-to-one map to the projective space \mathbb{P}^{Δ} , and we define Y_{Δ}^{+} to be the pullback of Y_{Δ} along this map. The sphere \mathbb{S}^{Δ} has homogeneous coordinates $(x_{\alpha} \mid \alpha \in \Delta \cap \mathbb{Z}^n)$, where we identify points that are proportional with a positive constant of proportionality. The group $\{\pm 1\}^{n+1}$ acts on \mathbb{S}^{Δ} where the last $((n+1)\text{st})$ coordinate acts as the antipodal map—global multiplication by ± 1 and the remaining coordinates $\{\pm 1\}^n$ act through the map φ_{Δ} (1.1),

$$(g, g_{n+1}) \cdot (x_{\alpha} \mid \alpha \in \Delta \cap \mathbb{Z}^n) = (g_{n+1} g^{\alpha} x_{\alpha} \mid \alpha \in \Delta \cap \mathbb{Z}^n).$$

The faces of Y_{Δ}^{+} are its intersections with coordinate subspaces \mathbb{S}^F of \mathbb{S}^{Δ} corresponding to faces F of Δ ,

$$\mathbb{S}^F := \{(x_{\alpha} \mid \alpha \in \Delta \cap \mathbb{Z}^n) \mid x_{\alpha} = 0 \text{ if } \alpha \notin F \cap \mathbb{Z}^n\}.$$

The isotropy subgroup of \mathbb{S}^F is

$$\{(g, g_{n+1}) \mid g_{n+1} g^{\alpha} = 1 \text{ for } \alpha \in F \cap \mathbb{Z}^n\}.$$

Vectors b in the normal cone σ_F to a face F of Δ have constant dot product with elements of F —define $b \cdot F$ to be this constant. Then the subgroup

$$\bar{\sigma}_F^{+} := \{(-1)^{(b, b \cdot F)} \mid b \in \sigma_F\} \subset \{\pm 1\}^{n+1}$$

is the isotropy group of \mathbb{S}^F , and therefore of the corresponding face of Y_{Δ}^{+} .

Proposition 1.3. *The spherical toric variety Y_{Δ}^{+} is obtained as the quotient of the cell complex $\Delta \times \{\pm 1\}^{n+1}$ by the equivalence relation*

$$(p, \xi) \sim (q, \eta) \iff p = q \text{ and } \xi \bar{\sigma}_F^{+} = \eta \bar{\sigma}_F^{+}, \text{ where } p \text{ lies in the face } F.$$

2. CHARACTERIZATION OF ORIENTABILITY

We characterize the orientability of a general (not necessarily smooth) small cover, and determine its number of components. Our elementary methods follow those of Nakayama and Nishimura [13]. We also establish the analogous results for spherical toric varieties.

Theorem 2.1. *Let $Y(\Delta, \mathcal{S})$ be a small cover of dimension n .*

- (1) $Y^\circ(\Delta, S)$ is orientable if and only if there exists a basis of $\{\pm 1\}^n$ such that each $\bar{\sigma} \in S$ for $\sigma \in P$ of rank 1 is generated by a product of an odd number of basis vectors.
- (2) The components of $Y^\circ(\Delta, S)$ are naturally indexed by $\{\pm 1\}^n / \langle \bar{\sigma} \mid \text{rank}(\sigma) = 1 \rangle$.

Thus $Y^\circ(\Delta, S)$ has 2^{n-k} connected components, where k is the rank of the subgroup $\langle \bar{\sigma} \mid \text{rank}(\sigma) = 1 \rangle$ of $\{\pm 1\}^n$.

Proof. For each $\sigma \in P$ with rank 1, let g_σ be the generator of $\bar{\sigma} \simeq \mathbb{Z}/2\mathbb{Z}$. Then $Y^\circ := Y^\circ(\Delta, S)$ is obtained by gluing (Δ, ξ) and (Δ, η) along F_σ whenever $\xi = \eta g_\sigma$ for some $\sigma \in P$ of rank 1, so the number of connected components of Y° is equal to the number of orbits of Y° under the action of $\langle \bar{\sigma} \mid \text{rank}(\sigma) = 1 \rangle$.

The space Y° is orientable if and only if its top integral homology group $H_n(Y^\circ, \mathbb{Z})$ is nontrivial. This group is the kernel $\ker \partial$ of the differential in the cellular chain complex of the cell complex Y° ,

$$C_n \xrightarrow{\partial} C_{n-1}.$$

Here C_n is the free abelian group generated by

$$\{\Delta\} \times \{\pm 1\}^n = \{(\Delta, \xi) \mid \xi \in \{\pm 1\}^n\}$$

and C_{n-1} is the free abelian group generated by

$$\{[F_\sigma, \xi] \mid \sigma \in P, \text{rank}(\sigma) = 1, \xi \in \{\pm 1\}^n\} / \sim,$$

where $[F_\sigma, \xi] \sim [F_\sigma, \eta]$ whenever $\xi \bar{\sigma} = \eta \bar{\sigma}$, which is $[F_\sigma, \xi] \sim [F_\sigma, \xi g_\sigma]$. Orient each facet F_σ so that

$$\partial(\Delta) = \sum_{\text{rank}(\sigma)=1} F_\sigma.$$

Consider an n -cycle

$$X = \sum_{\xi \in \{\pm 1\}^n} n_\xi \cdot (\Delta, \xi) \in C_n$$

on Y° , where $n_\xi \in \mathbb{Z}$. Then

$$\partial(X) = \sum_{\xi \in \{\pm 1\}^n} n_\xi \sum_{\text{rank}(\sigma)=1} [F_\sigma, \xi] = \sum_{\text{rank}(\sigma)=1} \sum_{\xi \in \{\pm 1\}^n / \langle g_\sigma \rangle} (n_\xi + n_{\xi g_\sigma}) [F_\sigma, \xi].$$

Hence an n -cycle X lies in $\ker \partial$ if and only if $n_\xi = -n_{\xi g_\sigma}$ for all ξ in $\{\pm 1\}^n$ and σ of rank 1. Equivalently, $n_\xi = (-1)^k n_{\xi g_{\sigma_1} \dots g_{\sigma_k}}$ for all $\xi \in \{\pm 1\}^n$ and σ_i of rank 1.

We show that $\ker \partial$ is non-trivial if and only if there exists a basis e_1, \dots, e_n of $\{\pm 1\}^n$ such g_σ is a product of an odd number of basis vectors, for each element $\sigma \in P$ of rank one. Let \mathbb{O} be the set of generators g_σ of $\bar{\sigma}$ for rank one elements $\sigma \in P$.

Suppose that there exists a basis e_1, \dots, e_n of $\{\pm 1\}^n$ such that each $g_\sigma \in \mathbb{O}$ is a product of an odd number of basis vectors. For $\xi \in \{\pm 1\}^n$ define n_ξ to be 1 if ξ is a product of an even number of the e_i and -1 if it is a product of an odd number of the e_i . Clearly, we have $n_\xi = -n_{\xi g_\sigma}$ for all ξ and σ , so $\ker \partial$ is non-trivial and hence Y° is orientable. Since the number of connected components is 2^{n-k} , the kernel is isomorphic to $\mathbb{Z}^{2^{n-k}}$.

If there is no such basis of $\{\pm 1\}^n$, then there is some $g_\sigma \in \mathbb{O}$ which is a product of an even number of other elements in \mathbb{O} , for otherwise we can reduce \mathbb{O} to a linearly

independent set and then extend it to a basis of $\{\pm 1\}^n$. We get $g_\sigma = g_{\sigma_1} \cdots g_{\sigma_{2k}}$ and hence $1 = g_\sigma g_{\sigma_1} \cdots g_{\sigma_{2k}}$, so for every ξ we get

$$n_\xi = (-1)^{2k+1} n_{\xi g_\sigma g_{\sigma_1} \cdots g_{\sigma_{2k}}} = -n_\xi,$$

which implies that $n_\xi = 0$ and hence $\ker \partial = 0$ and so Y° is non-orientable. 🗨️

We restate the orientability criteria of Theorem 2.1 for real toric varieties.

Theorem 2.2. *Let Y be a real toric variety defined by a fan Σ . Then Y° is orientable if and only if there exists a basis of $\{\pm 1\}^n$ such that $(-1)^v$ is a product of an odd number of basis vectors, for each primitive vector v lying on a ray of Σ .*

The condition of Theorem 2.2 is easily checked.

Lemma 2.3. *Given $A \subset \{\pm 1\}^n$, the condition that there exists a basis of $\{\pm 1\}^n$ such that each vector in A is a product of an odd number of basis vectors, is equivalent to the condition that no product of an odd number of vectors in A is equal to 1 in $\{\pm 1\}^n$.*

Proof. If we had $v_1 \cdots v_{2k+1} = 1$, then $v_{2k+1} = v_1 \cdots v_{2k}$, and expressing each v_i as the product of an odd number of basis elements of $\{\pm 1\}^n$ yields a contradiction. For the other implication, reduce A to a linearly independent set A' and then extend A' to a basis of $\{\pm 1\}^n$. If there were a vector in $A \setminus A'$ which is a product of an even number of vectors $v = v_1 \cdots v_{2k}$, we would have then had $v \cdot v_1 \cdots v_{2k} = 1$. 🗨️

We may check if the condition is satisfied by reducing A to a linearly independent set A' and checking if each vector in $A \setminus A'$ is a product of an odd number vectors in A' .

We omit the proof of the analog of Theorem 2.1 for spherical toric varieties, as there are essentially no differences.

Theorem 2.4. *Let $Y_\Delta^+ \subset \mathbb{S}^\Delta$ be a spherical toric variety defined by a full-dimensional lattice polytope $\Delta \subset \mathbb{R}^n$.*

- (1) *Y_Δ^+ is orientable if and only if there exists a basis of $\{\pm 1\}^{n+1}$ such that for each facet F of Δ with primitive normal vector b , the element $(-1)^{(b, b \cdot F)}$ is a product of an odd number of basis elements.*
- (2) *The components of Y_Δ^+ are naturally indexed by*

$$\{\pm 1\}^{n+1} / \langle (-1)^{(b, b \cdot F)} \mid b \text{ is a primitive normal vector to a facet } F \text{ of } \Delta \rangle.$$

3. EXAMPLES AND APPLICATIONS TO LOWER BOUNDS

We settle some questions of orientability left open in [18] and then explain the motivation for these results from the study of real solutions to systems of polynomials. We begin with an example.

3.1. Cross Polytopes. The cross polytope is the convex hull of the standard basis vectors e_1, e_2, \dots, e_n in \mathbb{R}^n and their negatives $-e_1, -e_2, \dots, -e_n$. When $n > 1$ the corresponding toric variety is singular. The rays of its normal fan are the vertices $(\pm 1, \pm 1, \dots, \pm 1)$ of the n -cube, and these all have the same image in $(\mathbb{Z}/2\mathbb{Z})^n$. Extending this common image to a basis of $(\mathbb{Z}/2\mathbb{Z})^n$ shows that the hypotheses of Theorem 2.1 hold. Thus the corresponding real toric variety is orientable and its smooth points have 2^{n-1} connected components. Figure 1 displays the cross polytope when $n = 2$ and an embedding in \mathbb{R}^3 of the corresponding real toric variety. This example was treated in detail in [19, § 7].

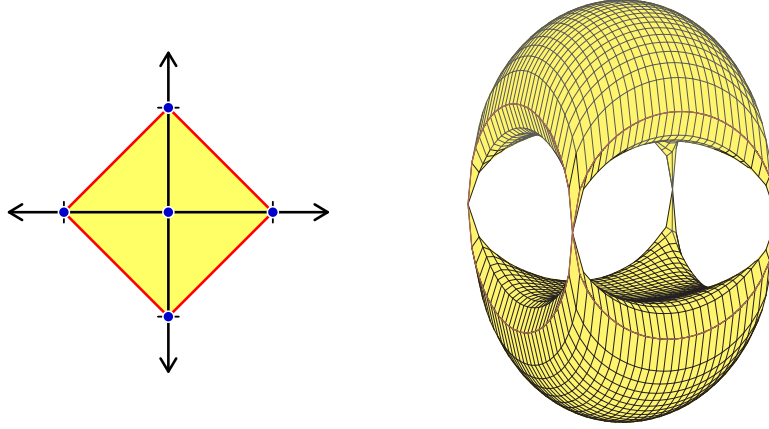


FIGURE 1. Two-dimensional cross polytope and double pillow.

3.2. Order Polytopes. The *order polytope* $O(P)$ [20] of a finite poset P is

$$O(P) := \{y \in [0, 1]^P \mid a \leq b \text{ in } P \Rightarrow y_a \leq y_b\}.$$

The integer points of $O(P)$ are its vertices and they correspond to the order ideals (upward-closed sets) of P .

Theorem 3.1. *The real toric variety $Y_{O(P)}$ is orientable if and only if all maximal chains of P have odd length.*

Proof. Lemma 4.9 of [18] (or rather its proof) implies that $Y_{O(P)}$ is orientable if all maximal chains of P have odd length. We establish the converse.

The order polytope has three types of facets

$$\begin{aligned} y_a &= 0 && \text{for } a \in P \text{ minimal,} \\ y_b &= 1 && \text{for } b \in P \text{ maximal,} \\ y_b - y_a &= 0 && \text{for } b \text{ covering } a \ (a < b) \text{ in } P. \end{aligned}$$

If we replace every $=$ by \geq , these each give a valid inequality for $O(P)$, which we write in matrix form as $\mathcal{A}y \geq c$. Then we have

$$(3.1) \quad O(P) := \{y \in \mathbb{R}^P \mid \mathcal{A}y \geq c\}.$$

By Theorem 2.2, $Y_{O(P)}$ is orientable if and only if there is a basis of the row space of \mathcal{A} , reduced modulo 2, such that each row is a sum of an odd number of basis vectors.

Fix a maximal chain $a_1 < \dots < a_k$ in P . The corresponding facets of $O(P)$ are

$$y_{a_1} = 0, \quad y_{a_2} - y_{a_1} = 0, \quad \dots, \quad y_{a_k} - y_{a_{k-1}} = 0, \quad y_{a_k} = 1,$$

and the corresponding rows of the matrix \mathcal{A} (reduced modulo 2) with the indicated columns are

$$\begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_{k-1} & a_k & & & \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \end{bmatrix}$$

Thus the sum of the rows that correspond to a fixed maximal chain of length k is equal to zero modulo 2, so we get a sum of $k + 1$ rows equal to zero. If k is even, Lemma 2.3 implies that $Y_{O(P)}$ is non-orientable. 🙄

A poset P is *ranked modulo 2* if all maximal chains in P have the same parity.

Theorem 3.2. *A spherical toric variety $Y_{O(P)}^+$ is orientable if and only if P is ranked modulo 2.*

Proof. Lemma 4.9 of [18] implies that $Y_{O(P)}^+$ is orientable if it is ranked modulo 2. We establish the converse.

Suppose that P is not ranked modulo 2. We exhibit an odd number of rows of the augmented matrix $[\mathcal{A} : c]$ whose sum is zero modulo 2, which will show that $Y_{O(P)}^+$ is not orientable, by Theorem 2.4 and Lemma 2.3, as these rows have the form $(b, b \cdot F)$ for b a primitive normal to a facet of the order polytope.

The order polytope is defined by the facet inequalities (3.1). For a maximal chain $a_1 \leq \dots \leq a_k$ in P , the corresponding rows of the augmented matrix $[\mathcal{A} : c]$ are

$$\left[\begin{array}{cccccccc|c} 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 & 1 \end{array} \right].$$

Observe that the sum of these rows is $[0 : 1]$. Note that each row of $[\mathcal{A} : c]$ has the form $(b, b \cdot F)$ (modulo 2), where b is a primitive normal vector to a facet F of Δ .

Since P is not ranked modulo 2, it has two maximal chains of different parities. Summing the rows of $[\mathcal{A} : c]$ which correspond to facets given by the two chains gives a sum of an odd number of rows of $[\mathcal{A} : c]$ which is equal to zero modulo 2. 🙄

3.3. Real solutions to systems of equations. In [18] we considered systems,

$$(3.2) \quad f_1(x_1, \dots, x_n) = f_2(x_1, \dots, x_n) = \dots = f_n(x_1, \dots, x_n) = 0,$$

where each f_i is a real polynomial whose exponent vectors lie in $\Delta \cap \mathbb{Z}^n$, for a fixed lattice polytope Δ , called the *Newton polytope* of the system. When the exponent vectors $\Delta \cap \mathbb{Z}^n$ affinely span \mathbb{Z}^n , the solutions to (3.2) may be expressed as a linear section $L \cap Y_\Delta$ of the real projective toric variety Y_Δ corresponding to Δ . Here $L \subset \mathbb{RP}^\Delta$ is a linear subspace

of codimension n . Projecting from a general codimension one linear subspace E of L , we may realize the solutions to (3.2) as the fibers of a map

$$\pi_E : Y_\Delta \longrightarrow \mathbb{RP}^n,$$

to real projective space. If n is odd, then \mathbb{RP}^n is orientable. If Y_Δ is also orientable, then fixing any orientation of \mathbb{RP}^n and Y_Δ , the map π_E has a degree whose absolute value gives a lower bound on the cardinality of a fiber of π_E , in particular on the number of real solutions to the system (3.2).

More generally, we may lift this projection to the spherical toric varieties

$$(3.3) \quad \pi_E^+ : Y_\Delta^+ \longrightarrow \mathbb{S}^n.$$

If Y_Δ^+ is orientable, we may fix an orientation and the absolute value of the degree of π_E^+ is a lower bound on the number of solutions to the system (3.2). Changing orientations in each component if necessary, we may assume that the degree is divisible by the number of components of $(Y_\Delta^+)^{\circ}$.

These discussions, as well as Theorems 2.2 and 2.4, have the following consequence for the existence of lower bounds for a system of polynomial equations.

Theorem 3.3. *Suppose that we have a system of polynomials (3.2) with Newton polytope Δ where $\Delta \cap \mathbb{Z}^n$ affinely spans \mathbb{Z}^n , and that we have expressed the system as a fiber of a projection map π_E^+ as in (3.3). If there is a basis for $\{\pm 1\}^{n+1}$ such that the element $(-1)^{(b, b \cdot F)}$ is a product of an odd number of basis elements for every primitive normal vector b to a facet F of Δ , then the absolute value of the degree of the map π_E^+ is a lower bound for the number of real solutions to (3.2), and this lower bound is a multiple of the number of components of $(Y_\Delta^+)^{\circ}$.*

Moreover, the map π_E^+ does not have a degree if this condition is not satisfied.

Remark 3.4. We did not need to consider the parity of n , for the condition of Theorem 2.2 implies that of Theorem 2.4. (A vector lies in a ray of the normal fan Σ to Δ if and only if it is normal to a facet F of Δ .)

3.4. Conclusions. We characterized the orientability of Y_Δ and Y_Δ^+ , which implies that the corresponding polynomial systems have lower bounds. The lower bound associated to a system of polynomials is the degree of a projection π_E or π_E^+ . For polynomial systems from posets [18], or more generally those corresponding to foldable triangulations [10, 18], these degrees have been computed. Our characterization of orientability replaces the condition in [18] that a variety is Cox-oriented and therefore strengthens the results of [18], in particular, it strengthens Theorem 3.5.

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